

Geo372
Vertiefung GIScience

Spatial Interpolation

Herbstsemester

Ross Purves

Last week

- We looked at **data quality and integration**
- We saw how uncertainty could be introduced by the **concepts** we chose, by the way we **measured and represented** these **concepts** and **during analysis**
- I emphasised the importance of **documenting data quality**, for example by using the **Spatial Data Transfer Standard**
- We looked at problems with overlay and in particular some characteristics of **slivers**
- We saw how **overlay** was used in some typical analysis operations and the **influence of data quality on the results**

Wo Mietpreis-Berge sich erheben

Von Iwan Städler, 8. September 2015 41 Kommentare »

Die höchsten Mieten zahlt man heute am Genfersee und in Zürich. Wie rasant sich das Wohnen in den vergangenen 15 Jahren in der Schweiz verteuerte, zeigen unsere animierte Grafik und eine interaktive Karte mit allen Gemeinden.

Teilen 352 | Twittern 182 | @Datenblog112 folgen 2.422 Follower

Stichworte:
Animation,
Interaktiv, Karte,
Mieten

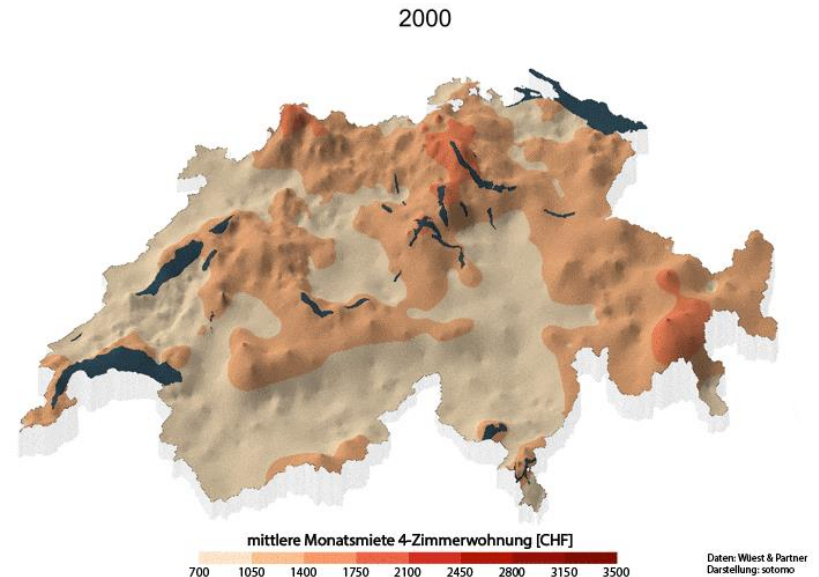
Für einmal stehen die markantesten Gebirge nicht in den Alpen. Stattdessen ragen sie bei Genf und Zürich in die Höhe. Auch in Zug und St. Moritz sind beachtliche Gipfel auszumachen. Willkommen in der Mietpreislandschaft Schweiz! Sie zeigt nicht geografische Höhen und Tiefen, sondern das Niveau der Mieten. Wo sich hohe Berge mit dunkelroten Gipfeln erheben, sind Wohnungen besonders teuer. In den hellen Talregionen hingegen darf man mit tiefen Mieten rechnen.

Anders als bei echten Bergen, deren Höhe sich über Jahrtausende kaum ändert, können Mietpreisberge rasant wachsen oder erodieren. Dies lässt sich hier eindrücklich beobachten:

[Tageanzeiger](#), 08.09.2015

Map by Mario Nowak, sotomo
Studied here, Masters in
GIScience

Today we will discuss how such
maps are produced



Outline

- What is **interpolation**, and when can we interpolate
- **Example interpolators** and their **properties**
- Describing **spatial variation** – introducing the **experimental semivariogram**
- **Formalising interpolation** through **regionalised variable theory**
- Kriging example – **ordinary kriging**

Learning objectives

- You can describe **spatial autocorrelation**, and understand why it is a prerequisite for **spatial interpolation**
- You can list key definitions in **interpolation** and illustrate interpolators that have such **properties** (e.g. exact/ approximate; global/ local; continuous/ abrupt)
- Given an **interpolator** and a **data set**, you can comment on the likely resulting **surface** and describe its key **features**
- Given a set of **sample data points**, you can illustrate how the **experimental semivariogram** is formed
- You can describe the key features of the **semivariogram model** (e.g. sill, range, nugget) and can sketch how it can be used in **geostatistical interpolation**

What is interpolation?

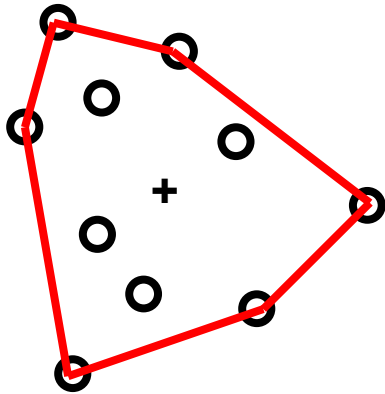
"...the process of predicting the value of attributes at unsampled sites from measurements made at point locations in the same area or region."

Burrough et al. (2015)

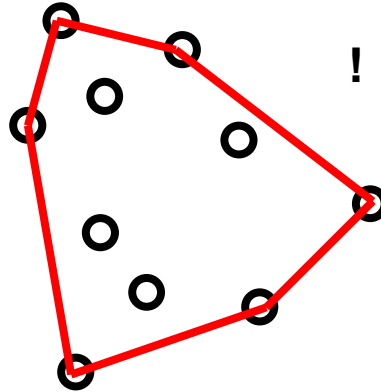
Some definitions

- **Interpolation vs. extrapolation** - interpolation estimates values within the convex hull of our data points/ extrapolation estimates outside the convex hull
- **Data points** – the samples we have available to interpolate with
 - **Support** – the area or volume of material used to sample (e.g. we might measure zinc concentration in a volume of 1m^3)
- **Sample points** – points at which we wish to estimate a value
- **Exact interpolator** – An interpolator which honours the data points (as opposed to an **approximate interpolator**)
- **Global interpolator** – Method where all the data points are used to estimate a field
- **Local interpolator** – Methods which use some subset of data points to locally estimate a field
- **Continuous** – smoothly varying field (with continuous values and, potentially, derivatives)
- **Abrupt** – a field whose values are discontinuous, or whose derivatives are discontinuous

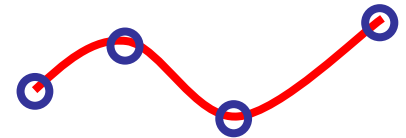
Definition sketches...



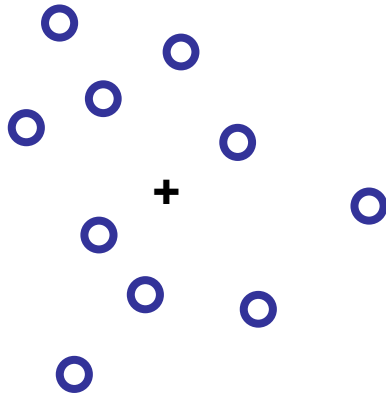
Interpolated point +



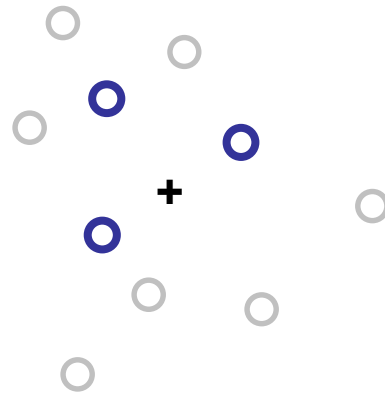
Extrapolated point !



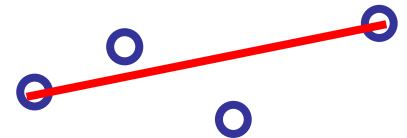
Exact interpolator



Global interpolator



Local interpolator



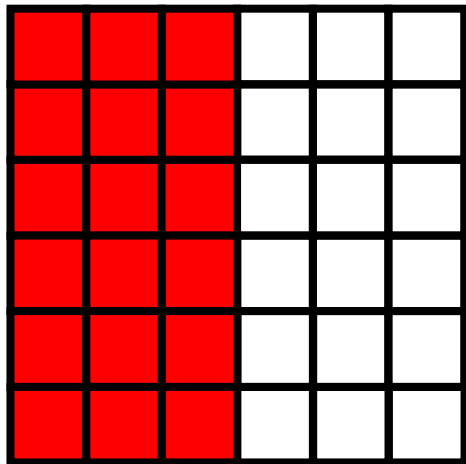
Approximate interpolator

Data points ●

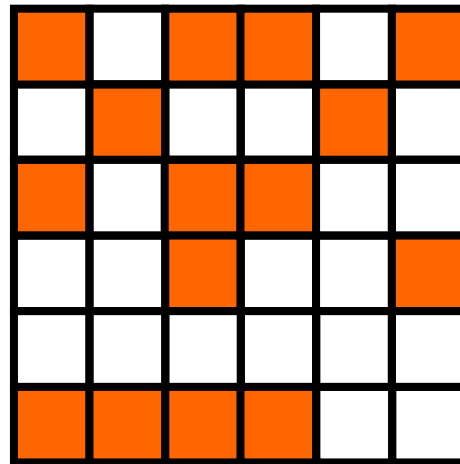
Tobler's First Law¹ (again)

“...everything is related to everything else, but near things are more related than distant things...”

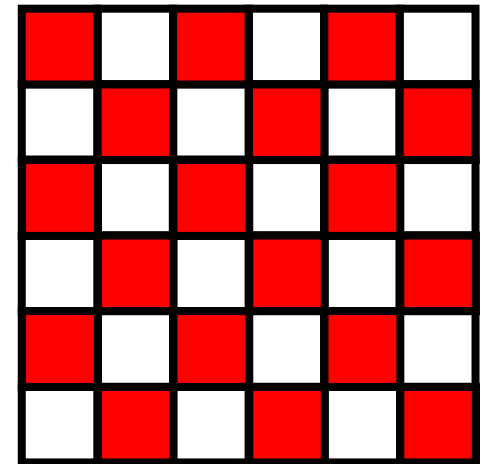
- In other words, many things that we measure in space are **spatially autocorrelated**



Positive autocorrelation



No autocorrelation

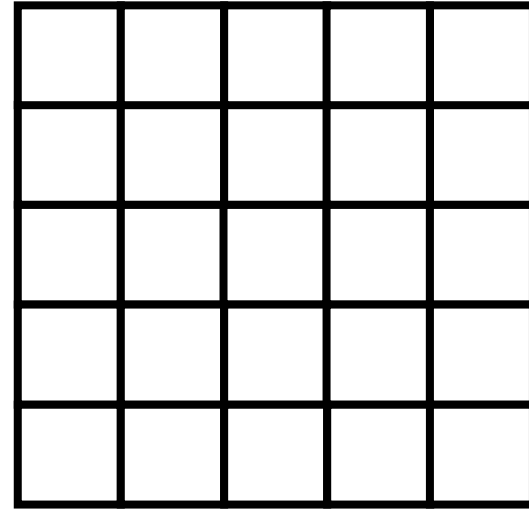


Negative autocorrelation

¹We could argue about the definition of a law, but we won't...

Spatial autocorrelation and interpolation

- Spatial autocorrelation is **implied** by Tobler's first law
- Interpolation **only** makes sense for properties which have some spatial autocorrelation
- If values are **distributed randomly**, we can guess values based on their distribution BUT **where** they are isn't important...



- Fill in the above grid by tossing a coin 25 times
- Compare the pattern with your neighbours and to the previous page
- Think of some **other examples** of attributes with such properties

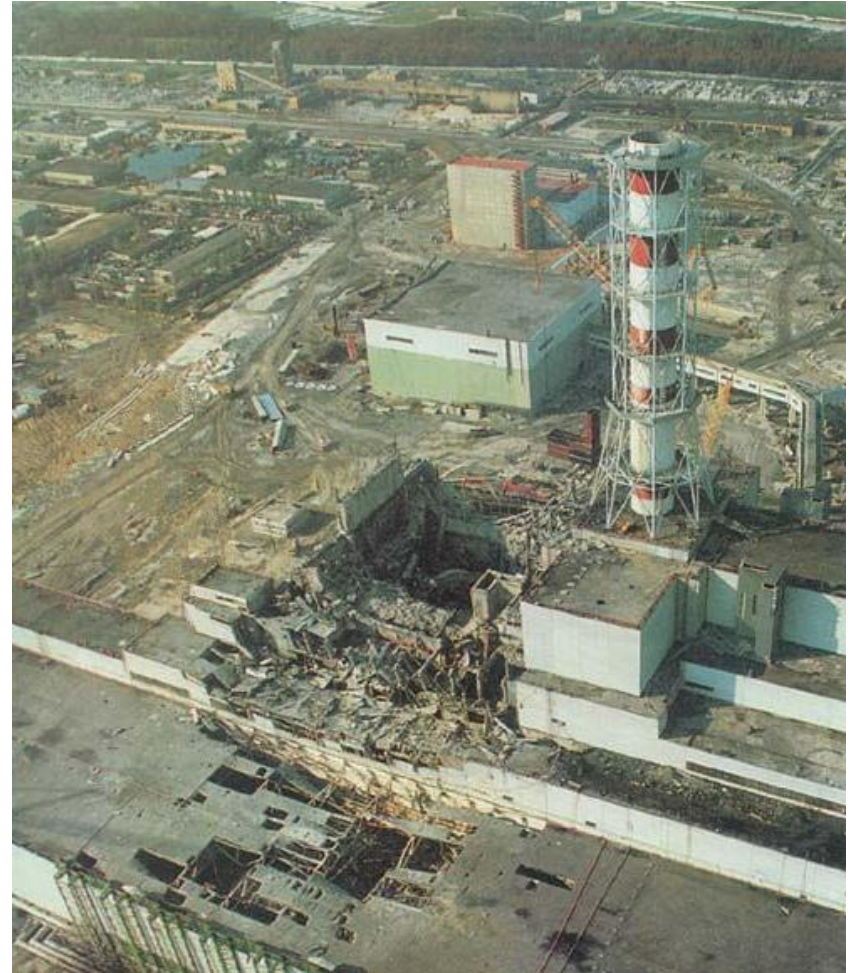
Why interpolate?

Burrough and McDonnell suggest 3 key reasons:

- when the data we have **do not completely cover** the area of interest
 - e.g. calculate a new **point or surface** from a set of input data points, lines or polygons
- when the discretised surface has a **different level of resolution, cell size or orientation** from that required
 - **Resampling** – for example when we wish to **overlay two rasters** with **different projections**
- when a continuous surface has a **different data structure** from that required
 - Typically **converting** a raster to vector or vice-versa

Example...

- On 26 April 1986 the **Chernobyl disaster** happened
- A **radioactive plume** spread across Europe
- One important predictor of radiation reaching the ground was **precipitation**
- Policymakers needed rapid information on **where** preventative measures needed to be taken
- Precipitation is already measured – can be used to suggest a **sampling strategy** for radiation measurements

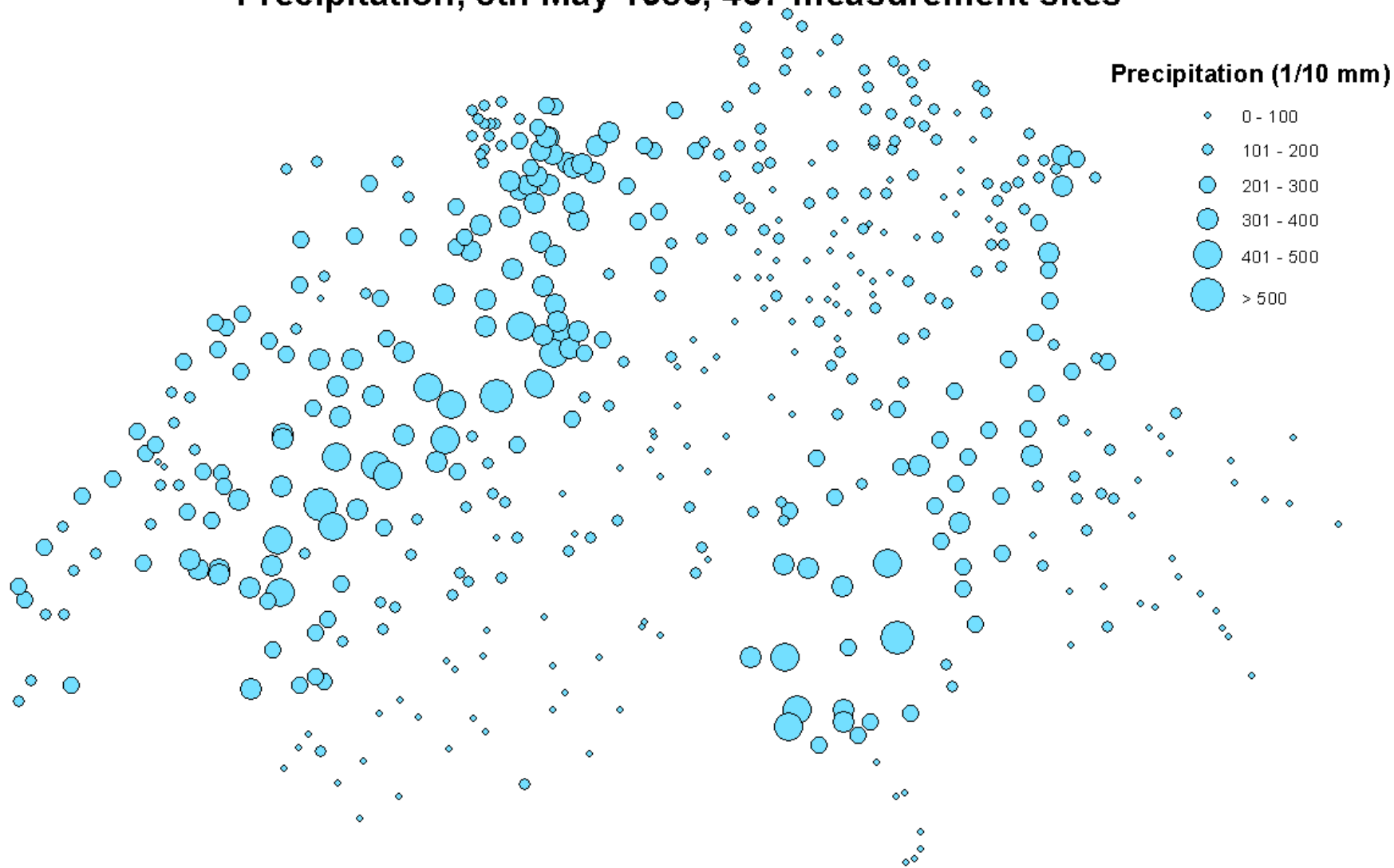


Source: http://en.wikipedia.org/wiki/Image:Chernobyl_Disaster.jpg

Example data set

- **Precipitation measurements** from Switzerland on 8th of May 1986 (post-Chernobyl)
- Total of **467 measurement locations**
- Measurement unit 1/10mm
- **25%** of these data were used for the **interpolations** I will show
- The other 75% were retained allow us to **calculate Root Mean Square Error**
- For all of the interpolations I interpolated a raster with a **resolution of ~1km**

Precipitation, 8th May 1986, 467 measurement sites



Global interpolation

- **Global interpolators** consider the values of all the data points
- They are a useful way of identifying **trends** in surfaces (large scale, systematic change)
- A trivial example of a trend would be **decreasing mean annual temperatures** as we moved north within Europe
- We attempt to identify such trends through **trend surface analysis**

Trend surface analysis

- For any **continuous field**, we can define a function: $z_i = f(x_i, y_i)$
- For a given form of function, f , we can **fit** a trend surface using **least squares**
- For this function we can then estimate **local residuals** (the difference between our data points and the surface)
- The simplest possible function (which has no trend) is the global mean: $z_i = \mathbf{constant}$
- The simplest trend surface is a **plane** through the points

Linear trend surface

Trend surface takes the form:

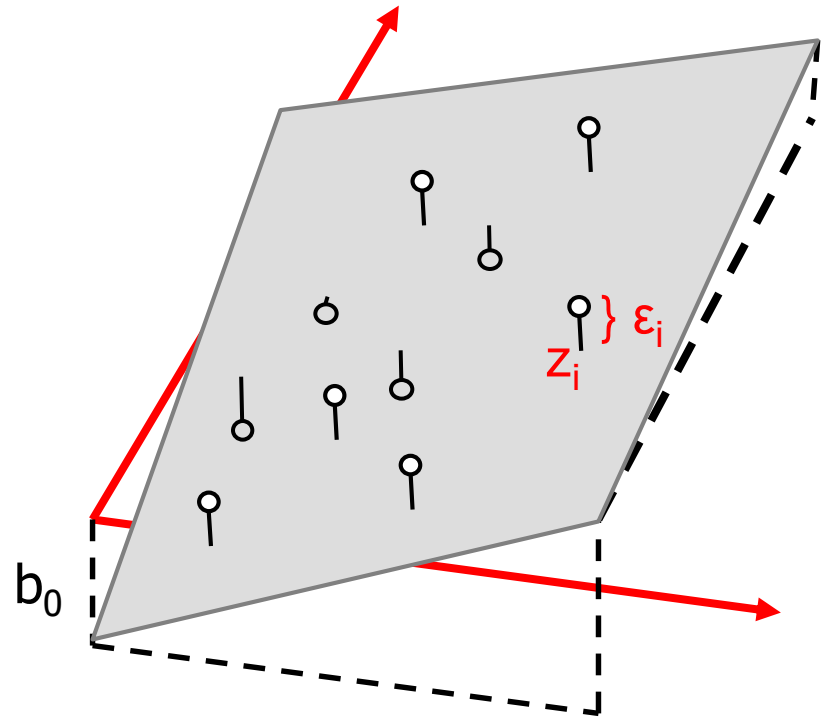
$$z_i = b_0 + b_1 x_i + b_2 y_i + \varepsilon_i$$

where b_0 = offset of surface at origin

b_1 = gradient in x-direction

b_2 = gradient in y-direction

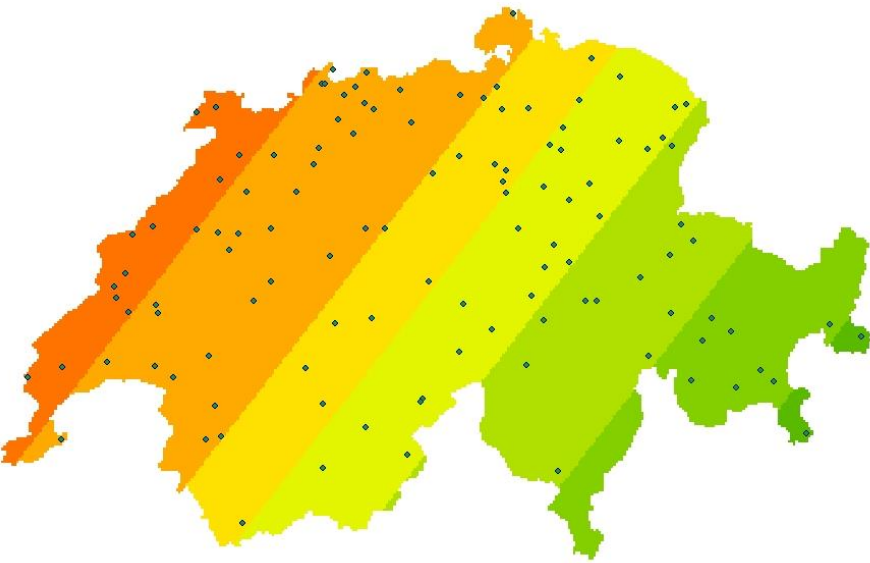
ε_i = residual at data point i



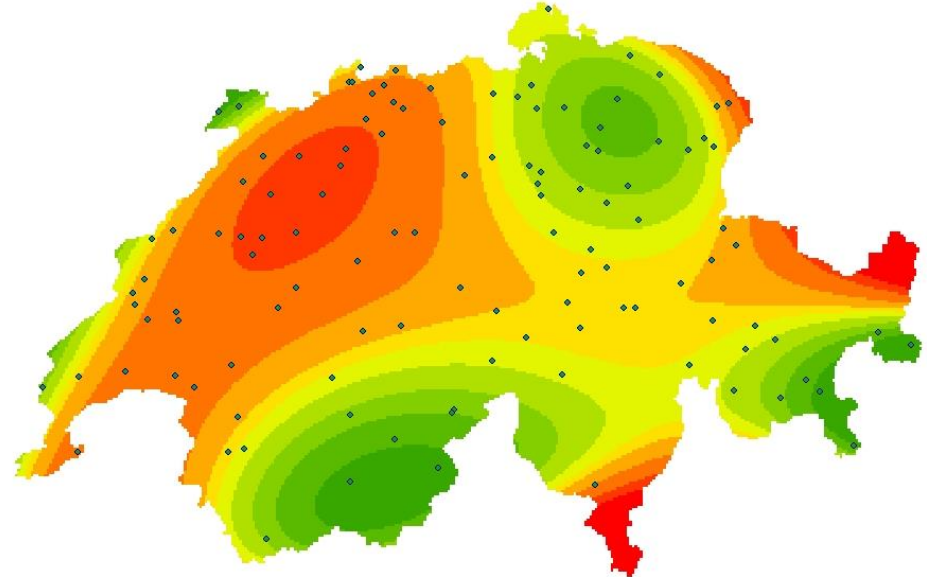
Having found a trend surface we can find r^2 (as for a linear regression)

We can extend our trend surface to any order of polynomial – but be careful...

Trend surface analysis examples



1st (linear) order trend surface



5th order trend surface

- Surfaces need not pass through the data points (**approximate interpolator**)
- **Higher order** models generally fit data better, but can have **large deviations**
- Can give a **very quick first look** at data (and identify, if we test for significance, trends to remove)
- Can you **describe a trend** in precipitation here (red-high/ green-low)?

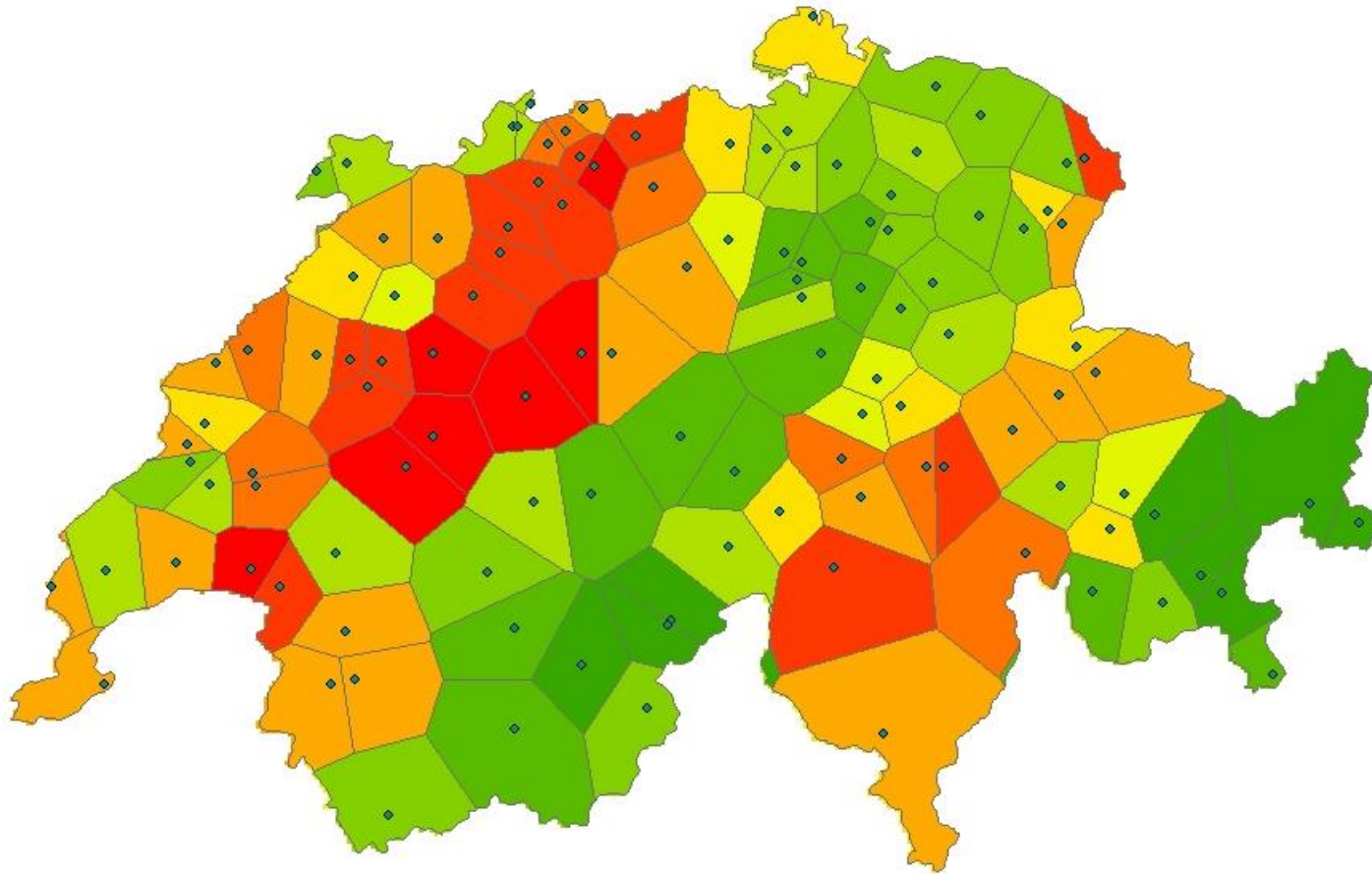
Local interpolation

- Local interpolators are all (implicitly) based on Tobler's First Law
- The key choice we have to make is **how** do we represent the idea of a thing being **near**
- You need to use your **knowledge of geography** to assess whether the representation of *near* is appropriate for the data we are interpolating
- You also need to know about the **properties of the interpolator** to assess whether the resulting surface is **realistic**

Thiessen polygons

- **Thiessen polygons** are the region which is nearest to a single point
- You covered deriving Thiessen polygons in GEO243
- We can interpolate by **assuming a value is constant** within the Thiessen polygons relating to a measurement point
- We are therefore saying: “sample points take the value of the nearest data points”

Thiessen polygon example



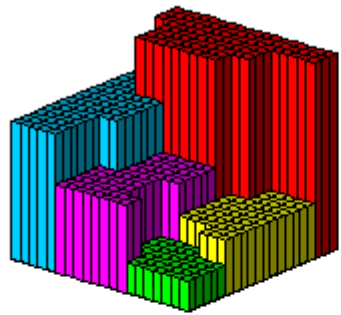
We can perform the same sort of interpolation by assigning a constant value everywhere in, for example, an administrative region

Pycnophylactic interpolation

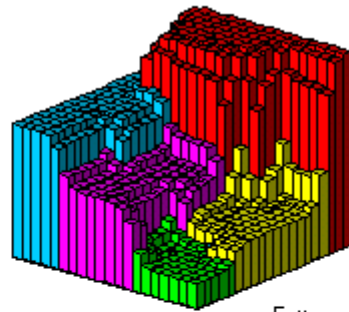
- Thiessen polygon's are an extreme case – we **assume homogeneity** within the polygons and **abrupt changes** at the borders
- This is **unlikely** to be correct – for example, precipitation or population totals don't have abrupt changes at arbitrary borders
- Tobler developed **pycnophylactic interpolation** to overcome this problem
- Here, values are reassigned by **mass-preserving reallocation**

Mass-preserving reallocation

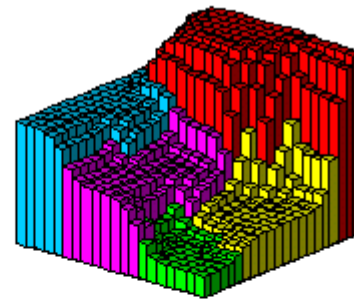
- The basic principle is that the **volume** of the attribute within a region **remains the same**
- However, it is assumed that a better representation of the variation is a **smooth surface**



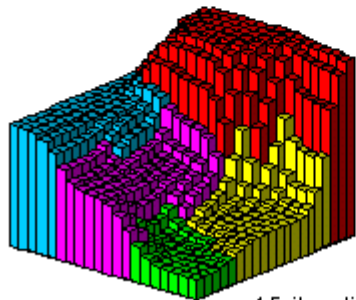
0 iterations



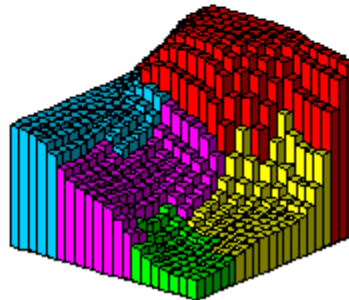
5 iterations



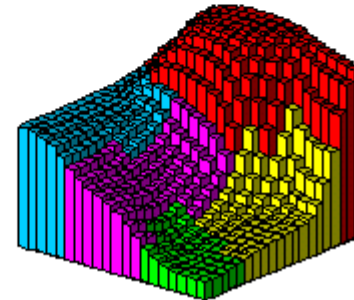
10 iterations



15 iterations



20 iterations



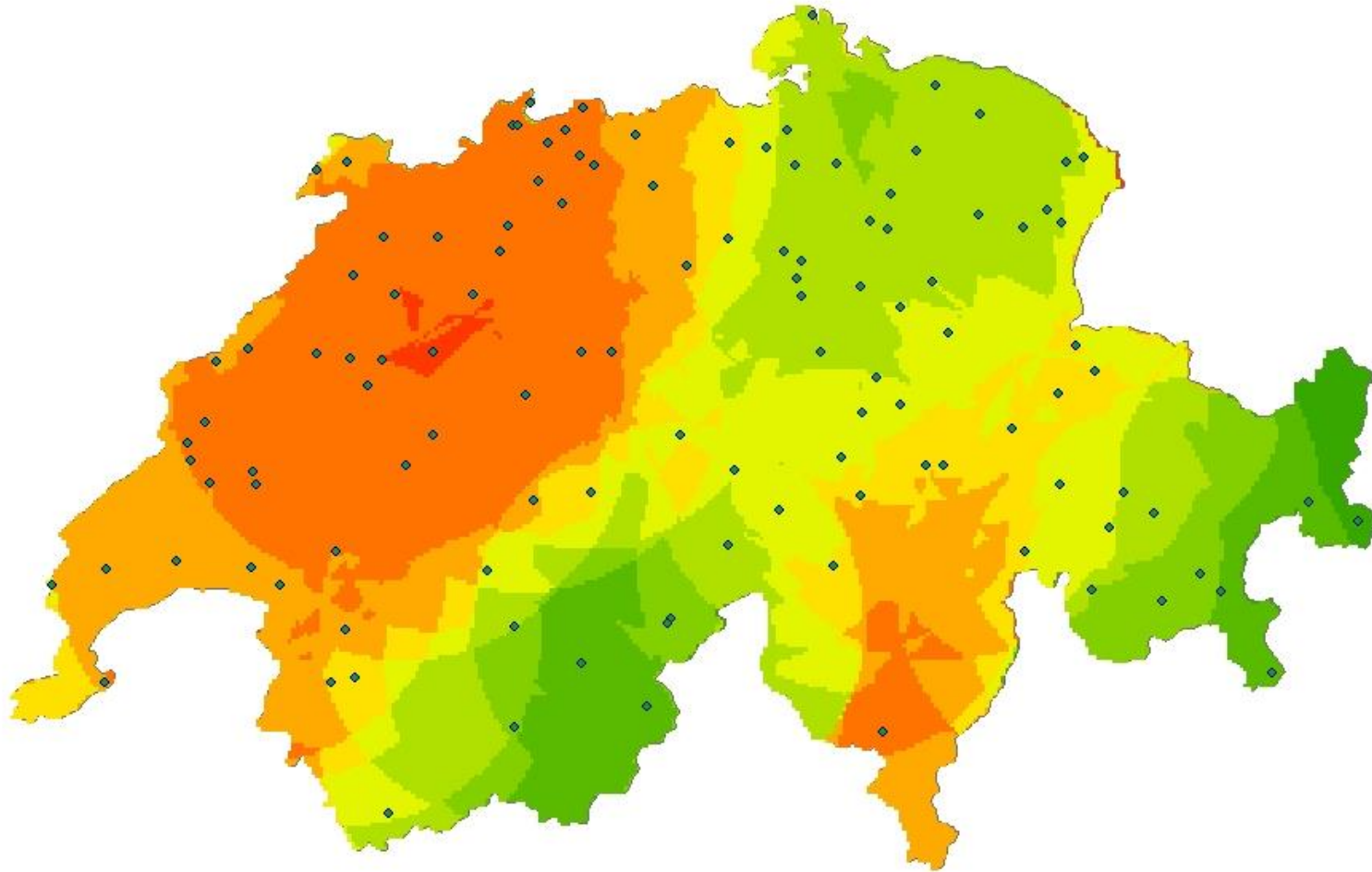
25 iterations

The volume (the sum within each coloured region) remains **constant**, whilst the surface becomes **smoother**. The solution is iterative – the stopping point is arbitrary.

Local spatial average

- Thiessen polygons use only the nearest point to interpolate a value
- The next “most simple” approach is to use the **mean of points** within a given **radius**, or a given **number of points**
- Trivial approach (effectively a **focal mean**)
- If no data points are within range, no value can be calculated
- **Inexact interpolator**, which produces surfaces with **abrupt discontinuities** (points jump in and out)

Local spatial average example



Local spatial average, radius 100km

Inverse distance weighting

- Inverse distance weighting explicitly takes account of nearness in two way:
 - Only points lying within some **kernel radius r** or some **fixed number n of near points** are used to calculate the mean
 - The points are also **weighted** according to their distance from the sample point

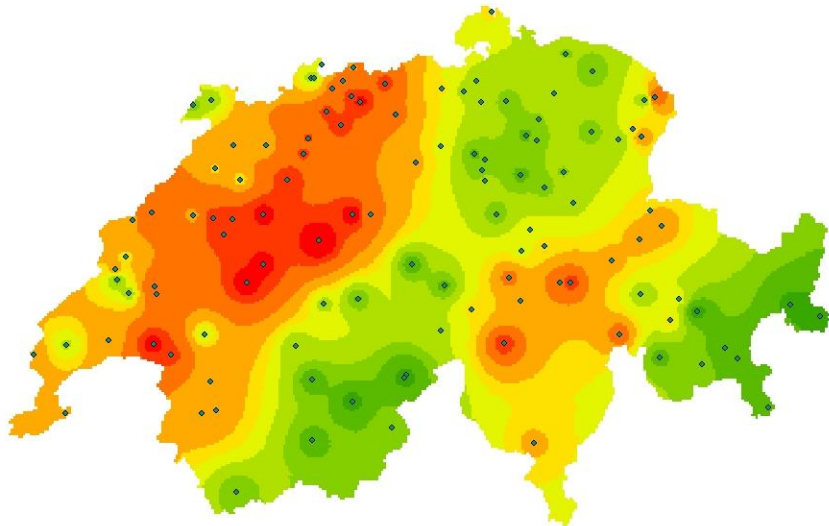
$$\hat{z}_j = \sum_{i=1}^m w_{ij} z_i \text{ where } w_{ij} \propto \frac{1}{d_{ij}^k}$$

here a group of m **data points** z_i contribute to sample point \hat{z}_j , w_{ij} is a **weight proportional** to the **distance** of the data point z_i from the sample point, and k is an **exponent**

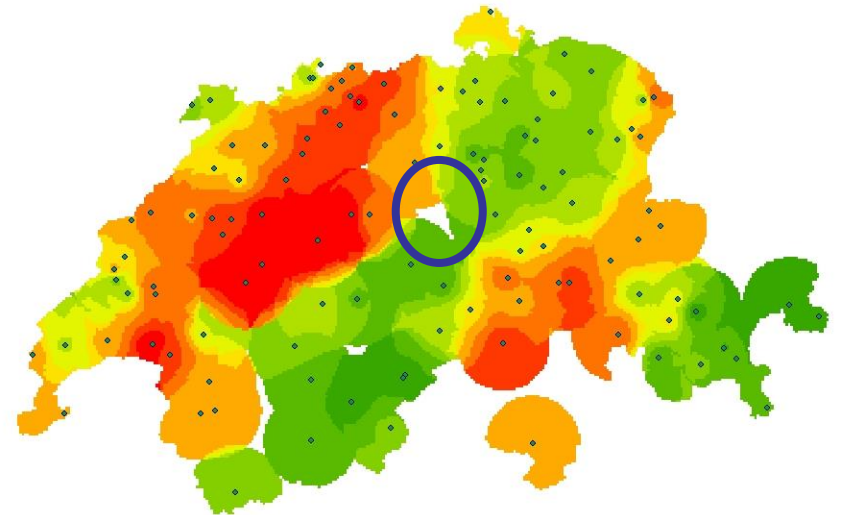
- At data points ($d_{ij}=0$), the sample point takes the value of the data point (IDW is an **exact interpolator**)

You already know all this!!

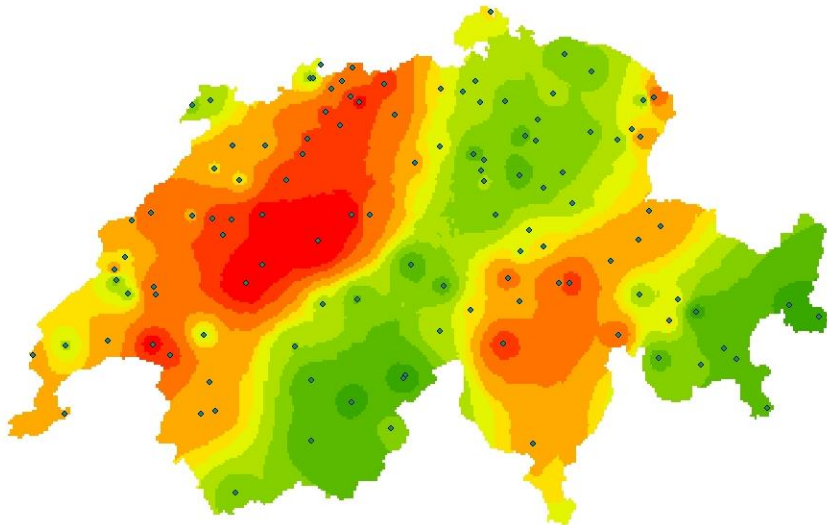
IDW examples



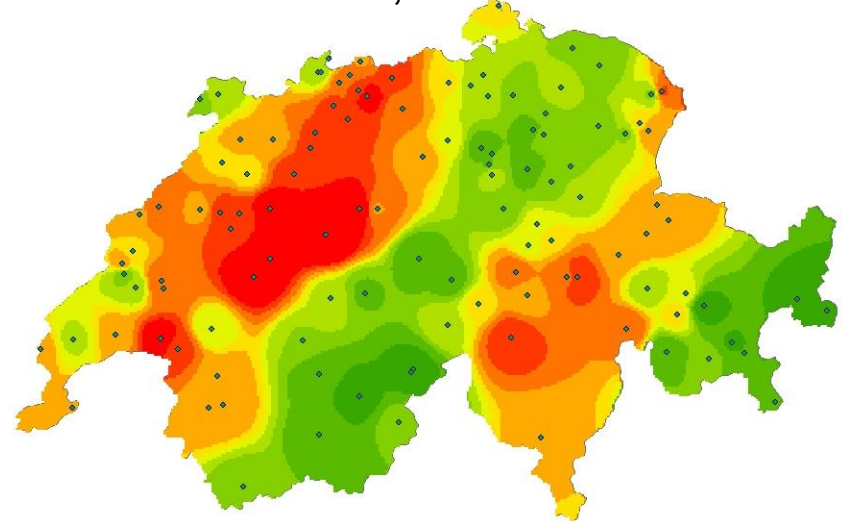
$r=100\text{km}$, $k=2$



$r=20\text{km}$, $k=2$



12 neighbours, $k=2$



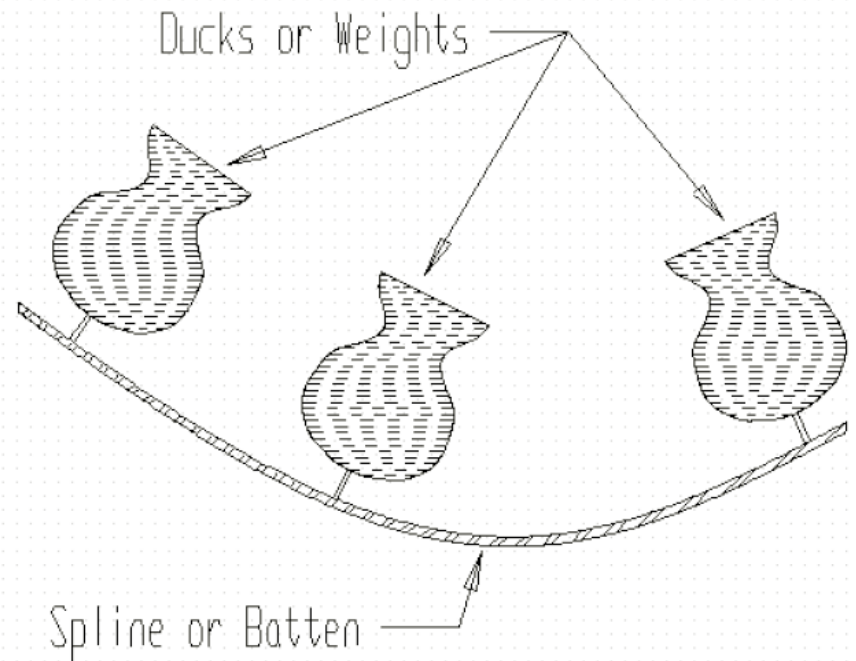
20 neighbours, $k=4$

Comments on IDW

- **Choice** of radius/ number of neighbours and distance weighting is **arbitrary**
- Form of surface is **strongly dependent** on these parameters (e.g. compare “bulls-eyes” for $k=2$ with $k=4$)
- Values of surface cannot be lower/ higher than the data
- Surface **varies smoothly** (except in case where radius is too small)

Splines

- **Splines** were originally used by draughtsmen to **locally fit smooth curves** by eye (particularly in boat building)
- They used flexible rulers, which were weighted at data points
- These curves are **approximate to piecewise cubic polynomials**, and have 1st and 2nd order continuity



Defining splines

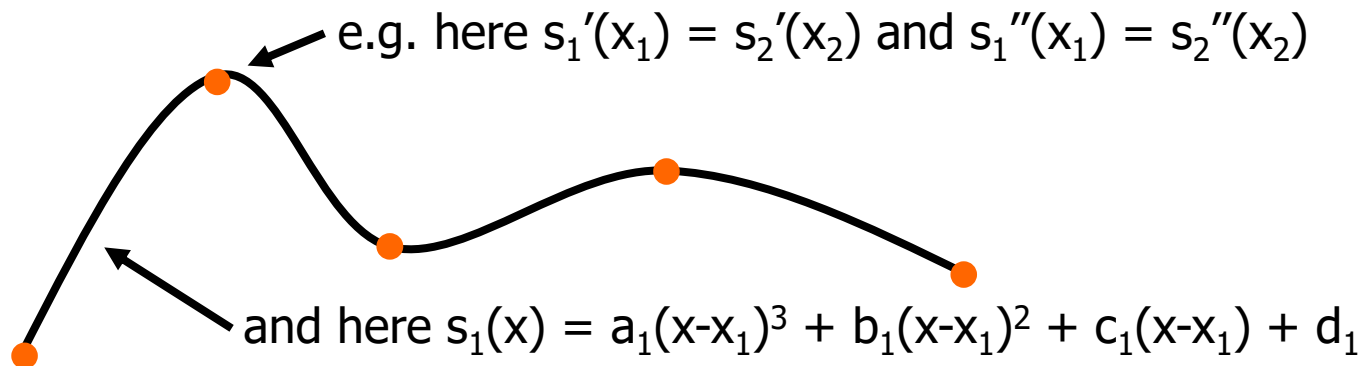
Splines interpolate through a series of **piece-wise polynomials** (s_i) between data points and aim to **minimise curvature**, thus for the interval $[x_i, x_{i+1}]$ a cubic spline takes the following form:

$$s_i(x) = a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i$$

for $i = 1, 2, \dots, n-1$

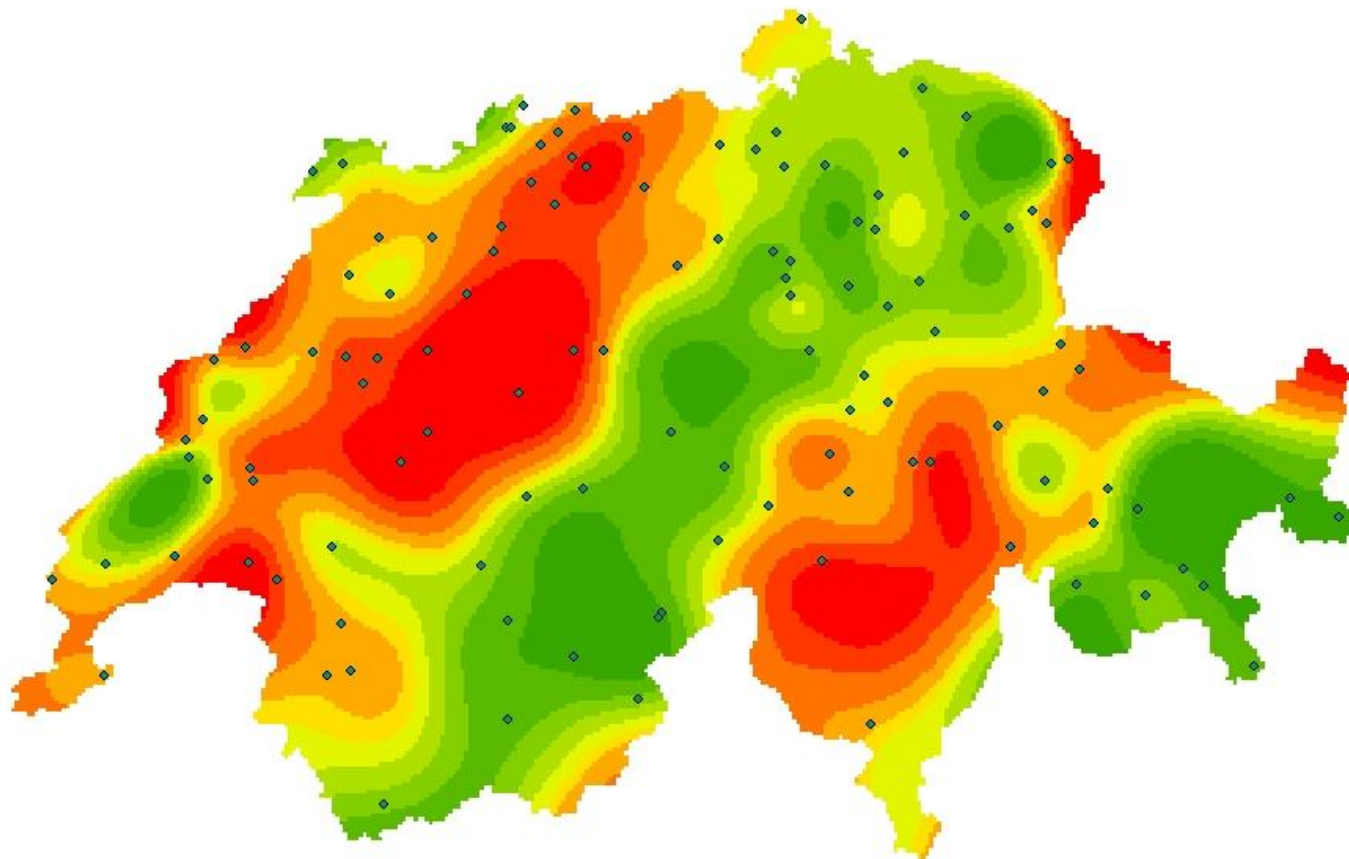
where

$$s_i'(x_i) = s_{i+1}'(x_{i+1}) \text{ and } s_i''(x_i) = s_{i+1}''(x_{i+1})$$



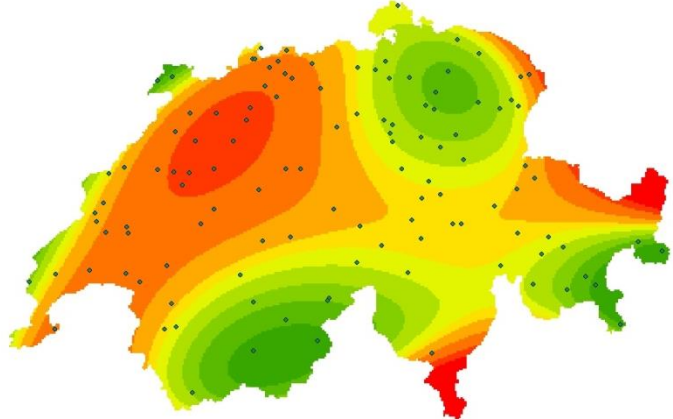
For a surface we use **bi-cubic splines**, but the principles are the same

Spline example

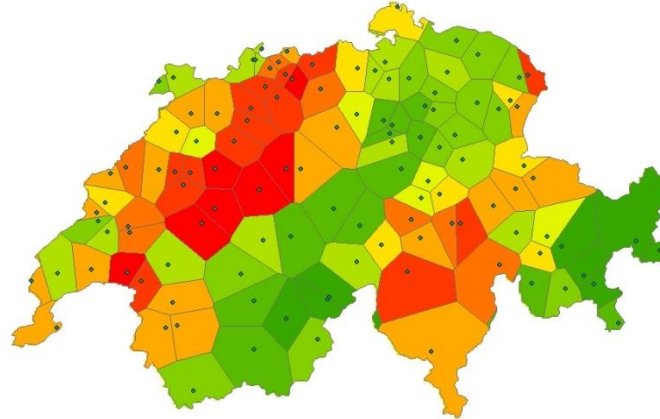


Comments on splines

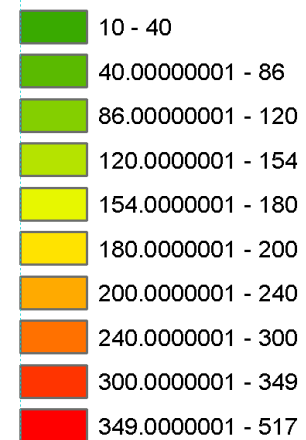
- Splines produce very **visually pleasing** surfaces (1st and 2nd order continuity)
- Because splines are piece-wise they are very **quick to recalculate** if a single data point changes
- Not all data are in reality so smooth – think about terrain
- **Maxima and minima** may be greater or less than the data values



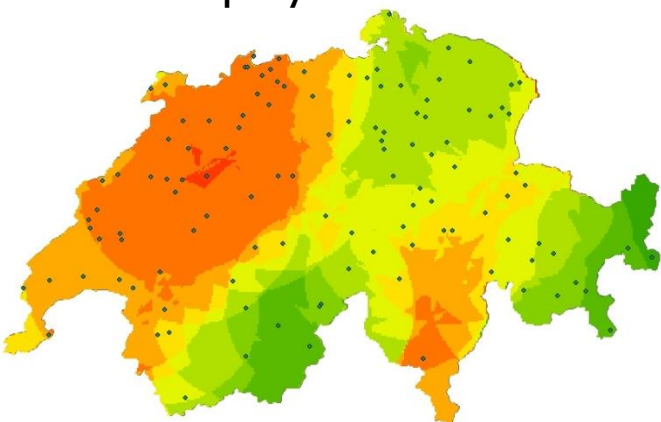
5th order polynomial RMSE=16.0mm



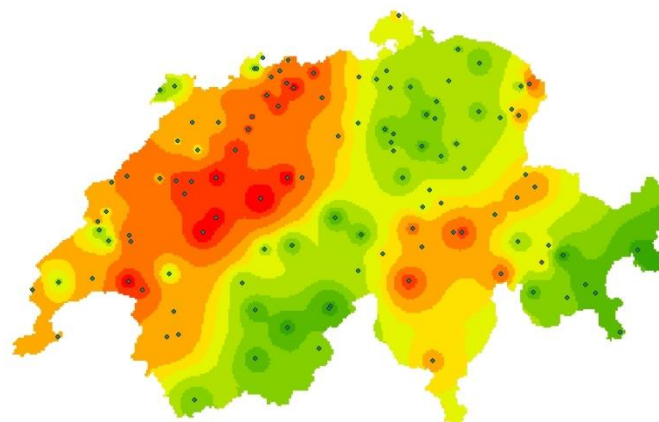
Thiessen RMSE=7.6mm



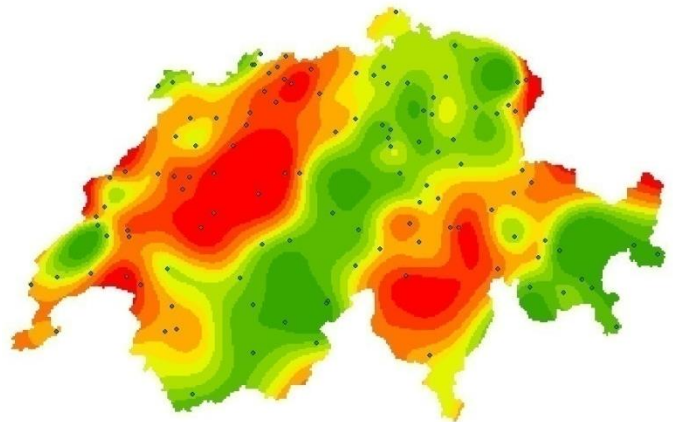
Precip (1/10mm)



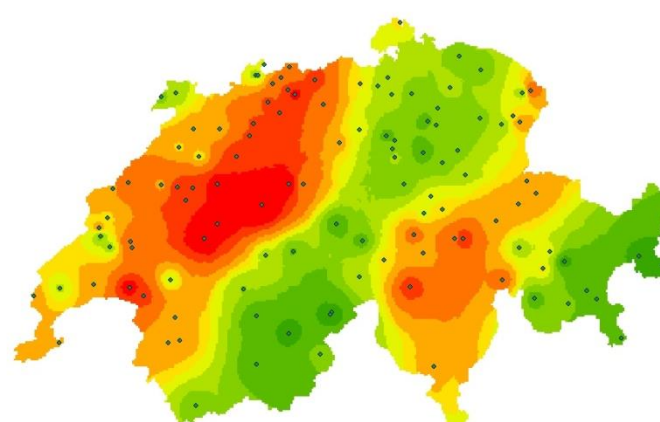
Focal mean RMSE=8.6mm



IDW(r=100km,k=2) RMSE=7.5mm



Splines RMSE=7.8mm



IDW(12 neighbours,k=2) RMSE=6.7mm

Which interpolator should I use?

That depends:

- Does the field you are trying to model **vary smoothly** across space?
- Should the field you are trying to model have **continuous 1st/2nd derivatives**?
- Should the **values** at the data points be **preserved**?
- How do you think values vary as a **function of distance**?
- Could the data have some **global trend**?
- Do you have enough data to **withhold** some for validation (unusual – data points are expensive)
- Are there any secondary data we can use to **constrain the interpolation**?

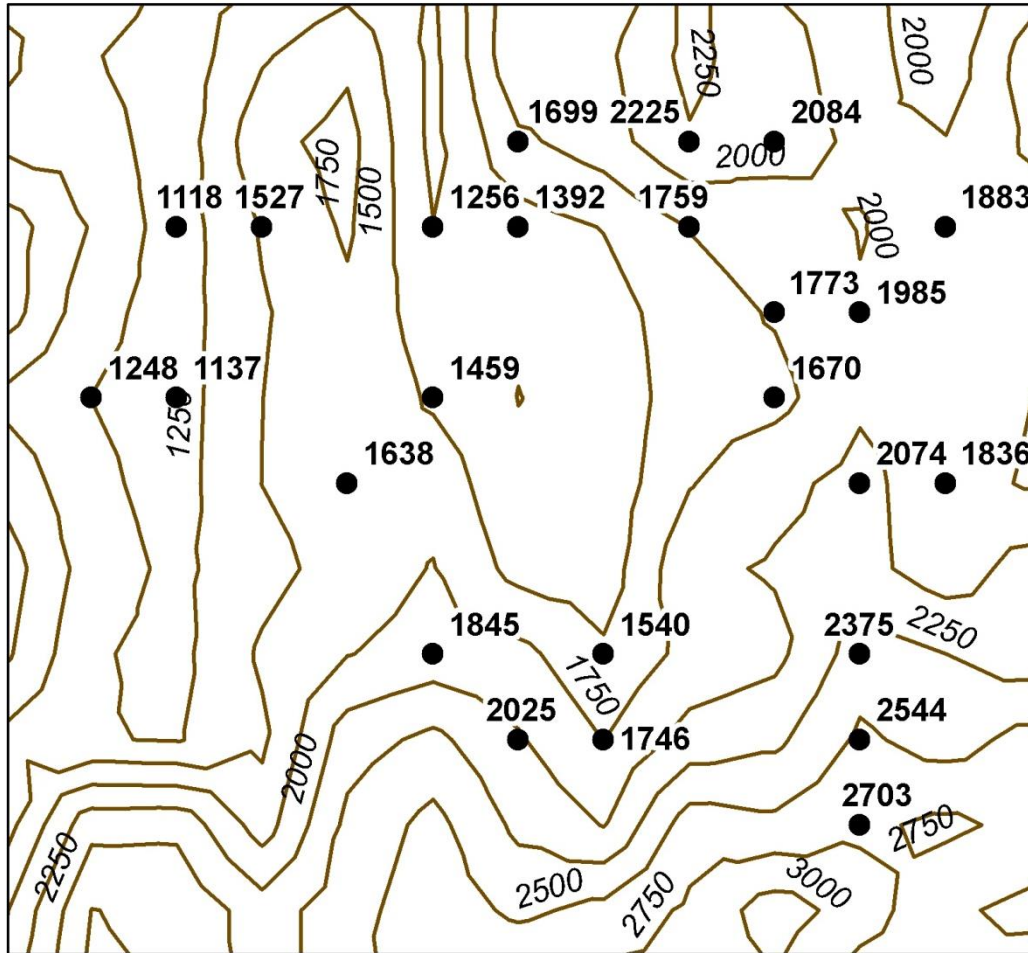
Improving IDW

- In general, most of the **local interpolators** provided some **not unreasonable** estimation of our surface
- We got the best(?) results (in terms of RMSE) for IDW
- But our **choice** of radius and distance weighting was arbitrary
- A key question is thus, can we use **theory** to inform these choices better?
- **Geostatistical techniques** are embedded in theory and aim to guide us in this decision making process

Back to Tobler's First Law

- **Geostatistical techniques** (and all interpolation) are underpinned by spatial autocorrelation
- The key notion here is that we expect attribute values close together to be more similar than those far apart
- We can **represent** this by **graphing** the **square root of the difference between attribute pair values** against the **distance by which they are separated...**
- This representation is known as the variogram cloud

Sample data



Contour interval 250m

20 randomly sampled
values (from 1km
raster)

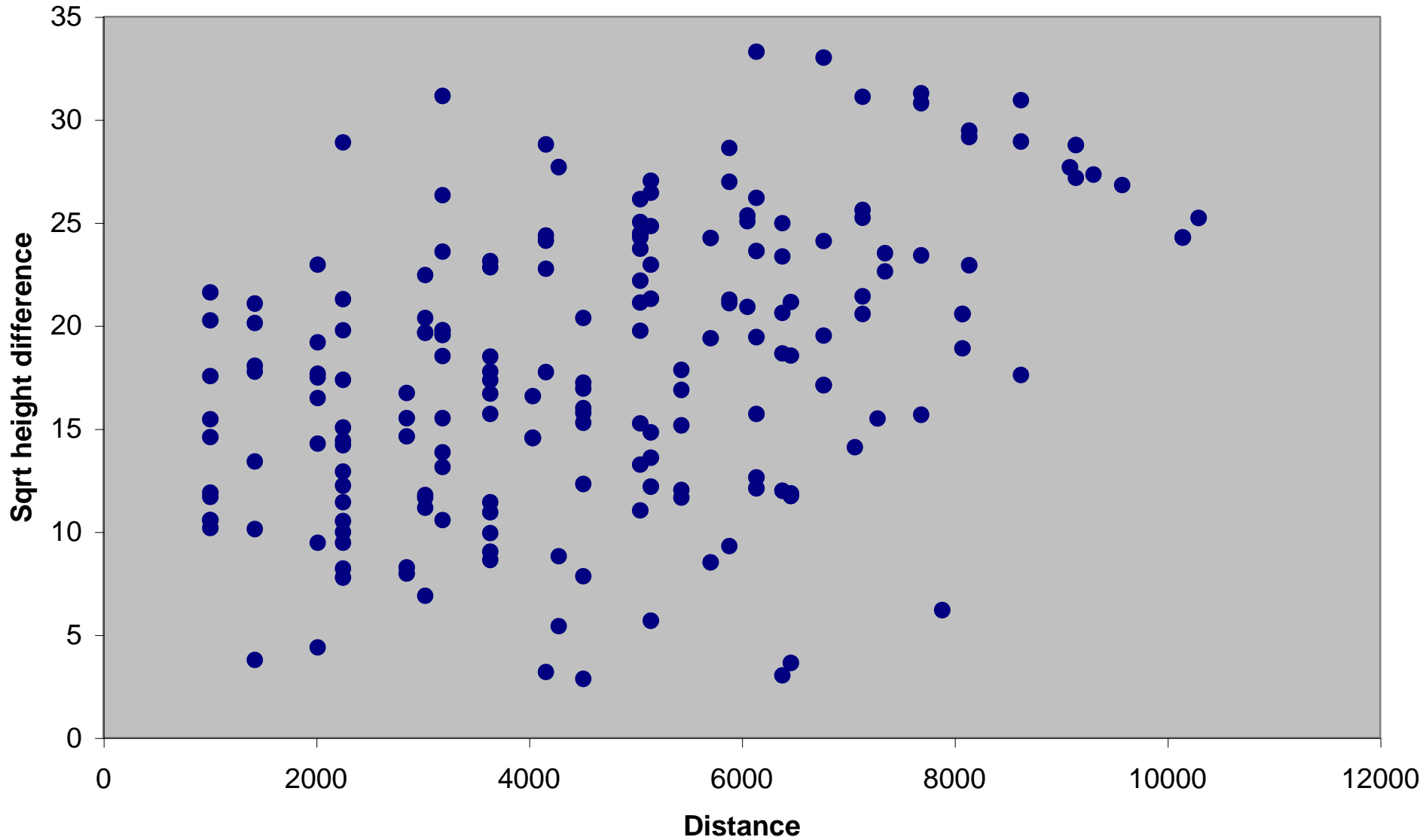
Question: Do these
data contain a **trend**?

Can you describe it in
words?

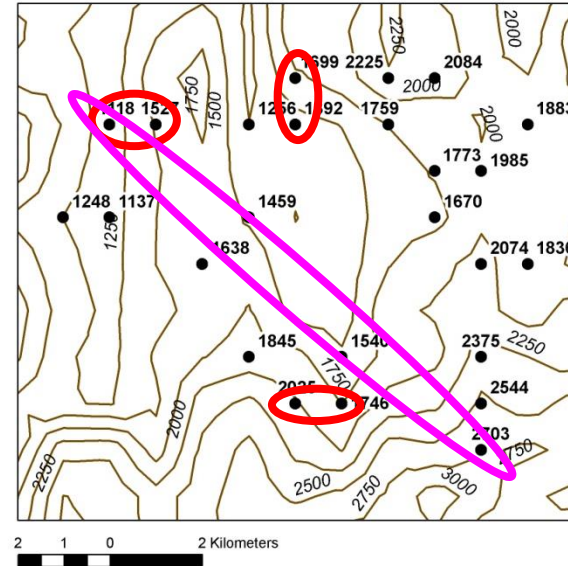
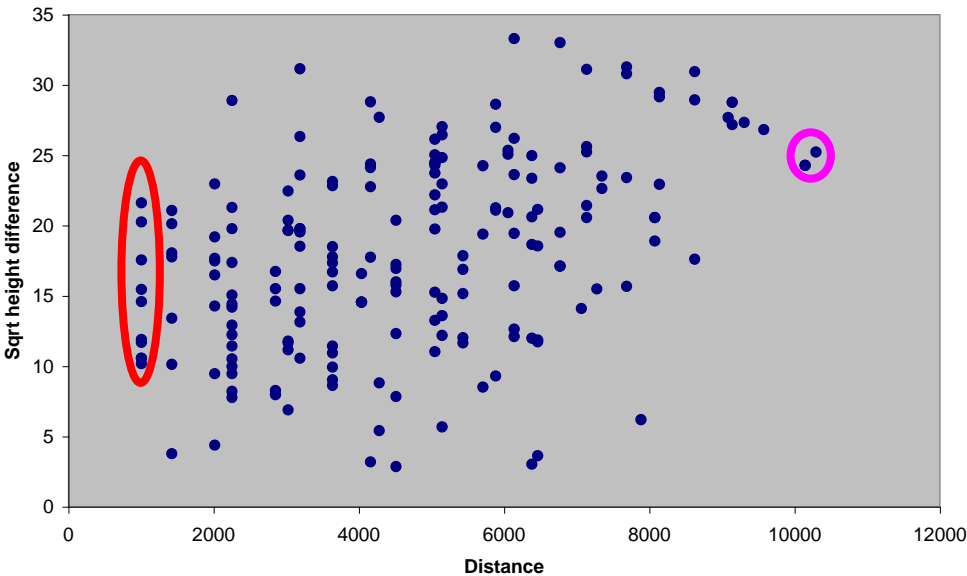
2 1 0 2 Kilometers



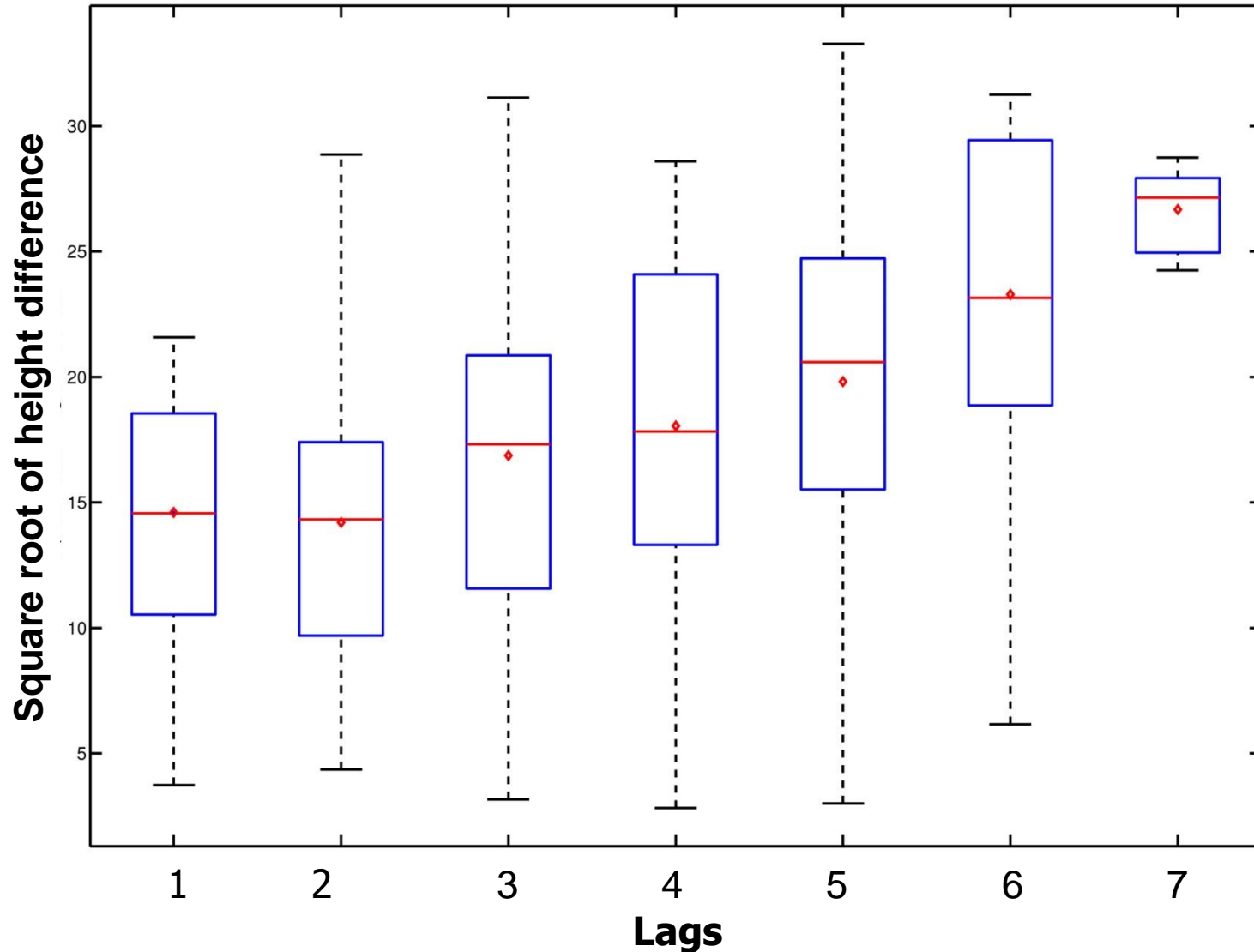
Variogram cloud



Understanding the variogram cloud



- We can see that **Tobler's first law** seems to be **true** for these data – points further away seem to be more different
- But already, for only 20 points we have 191 pairs - the semivariogram cloud is **hard to interpret**
- One way to see better what is happening is to **summarise** the semivariogram cloud

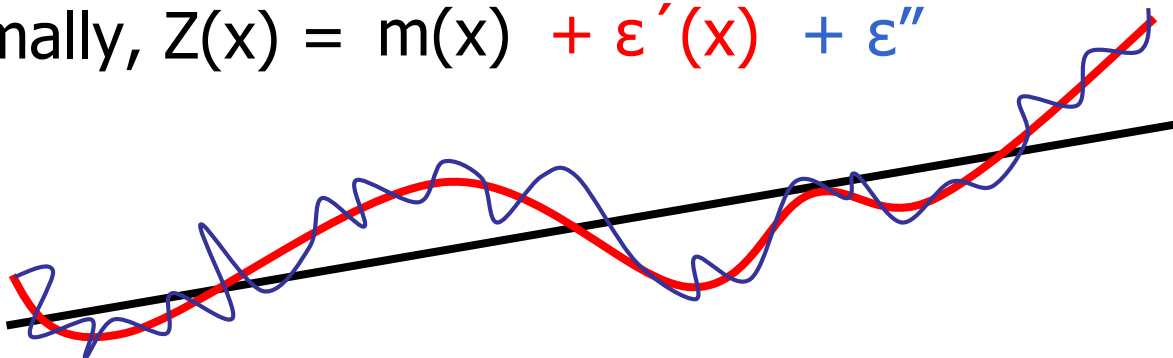


These box plots show the median and mean value for **lags** (e.g. 0-1500m; 1500-3000m etc.). Box boundaries show lower and upper quartiles and whiskers show extremes.

This plot nicely summarises our variogram cloud – what is happening to **height difference** with **increasing lag**?

Regionalised variable theory (1)

- **Regionalised variable theory** (RVT) attempts to formalise what we have seen up to now
- RVT says that the value of any **variable varying in space** can be expressed, at a location x , as the sum of:
 - The **structural component** (e.g. a constant mean or trend $m(x)$).
 - The **regionalised variable**, a random, but spatially autocorrelated component (that is the variation which we model with a local interpolator ($\varepsilon'(x)$))
 - A spatially uncorrelated **random noise component** (variability which is not dependent at all on location ε'')
- Formally, $Z(x) = m(x) + \varepsilon'(x) + \varepsilon''$



Regionalised variable theory (2)

- For these 3 components, the first, $m(x)$, can be modelled by a **global trend surface** (or the mean)
- The last ε'' is **random** and not related to location
- The second, the **regionalised variable**, describes how a **variable varies locally** in space
- This relationship can be described by the **experimental semivariance**:

$$\hat{\gamma}(h) = \frac{1}{2n} \sum_{h_{ij}=h-\Delta/2}^{h+\Delta/2} (z_i - z_j)^2$$

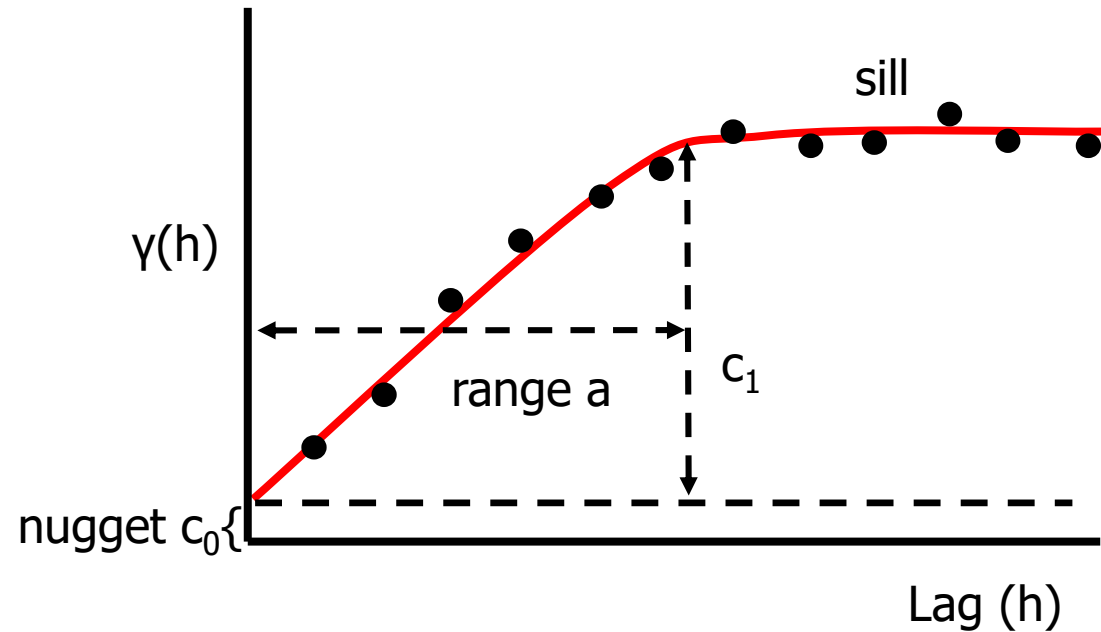
where h is the distance between a pair of points, z is the value at a point, n is the number of points, and Δ is the lag width

- This is exactly how we summarised our variogram cloud with the **box plot**...

Experimental semivariogram

The **experimental semivariogram** plots the **semivariance** and has a number of key features:

- **Nugget** (c_0): effectively describes uncertainty or error in attribute values (equivalent to ε'')
- **Sill** ($c_0 + c_1$): the value where the curve becomes level– in other words the variance between points reaches a maximum at this separation
- **Range** (a): the distance over which differences are spatially dependent



Black points show **experimental semivariance** for a range of lags, **red line** shows a **curve fitted** to them

Modelling the semivariogram

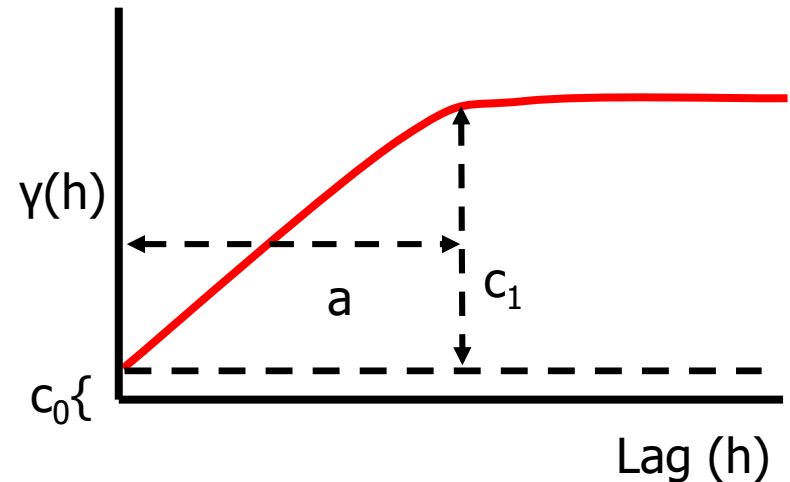
- The experimental semivariogram can take a range of forms
- One classic example is the **spherical model**:

$$\gamma(h) = c_0 + c_1 \left[\frac{3h}{2a} - 0.5 \left(\frac{h}{a} \right)^3 \right]$$

where $h < a$, and where $h \geq a$

$$\gamma(h) = c_0 + c_1$$

- What happens here when **$h \sim a$** ?



- Other models include Gaussian, exponential, etc...
- **Fitting** the semivariogram model is an **art!**

Ordinary Kriging

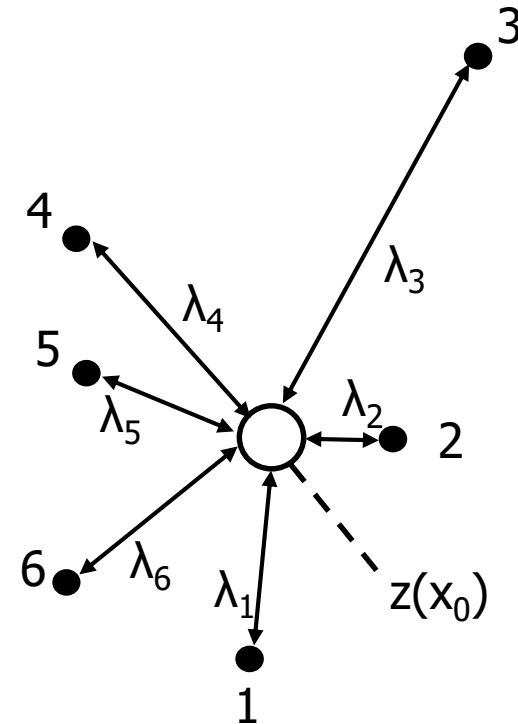
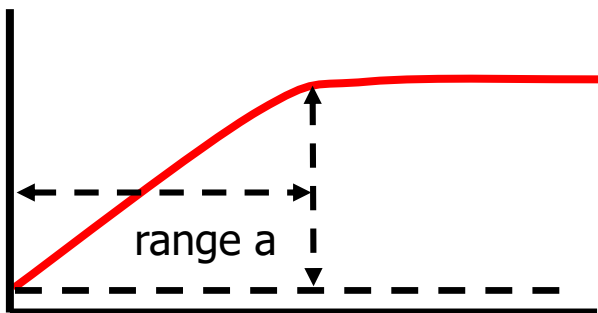
- Kriging uses the semivariogram that we fit to our data to **intelligently interpolate**
- Kriging is another **local interpolator**, which weights the data points according to distance
- There are many sorts of kriging – we will look here briefly at **ordinary kriging**
- We must make **several assumptions**
 - Surface has **no trend**
 - Surface is **isotropic** (variation is not dependent on direction)
 - We can **define a semivariogram** with a simple mathematical model
 - The same **semivariogram** applies over the **whole area** – all variation is a function of distance (this is known as the **intrinsic hypothesis**)

Ordinary kriging

- The value $z(x_0)$ is estimated as follows:

$$z_0(x) = \sum_{i=1}^n \lambda_i \cdot z(x_i)$$

- Similar to IDW, but **weights** are **chosen intelligently** by inverting the variogram model
- When we reach the range a , $\lambda_i = 0$



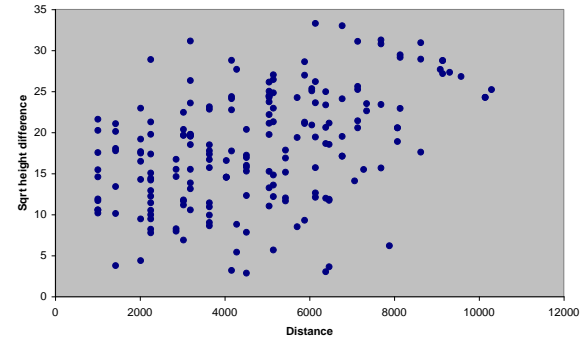
z_1, z_2, z_4, z_5 & z_6 contribute to the value of $z(x_0)$.

Their **weights** ($\lambda_1, \lambda_2, \dots$) are a function of the **variogram model**.

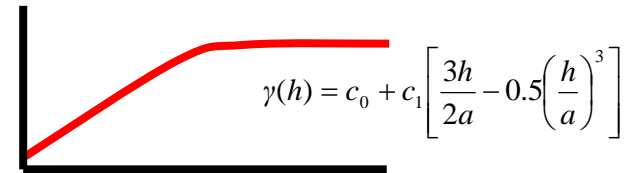
Because z_3 is further away from $z(x_0)$ than the range (a) its weight (λ_3) is **zero**

Typical steps in kriging

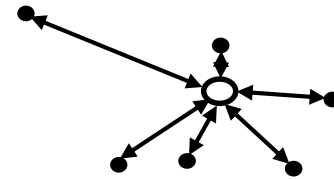
1. **Describe** spatial variation in data points (no spatial autocorrelation – no interpolation)



2. **Summarise** this variation through a mathematical function (Hard – lots of practise required)



3. Use this function to **calculate weights** (Kriging)

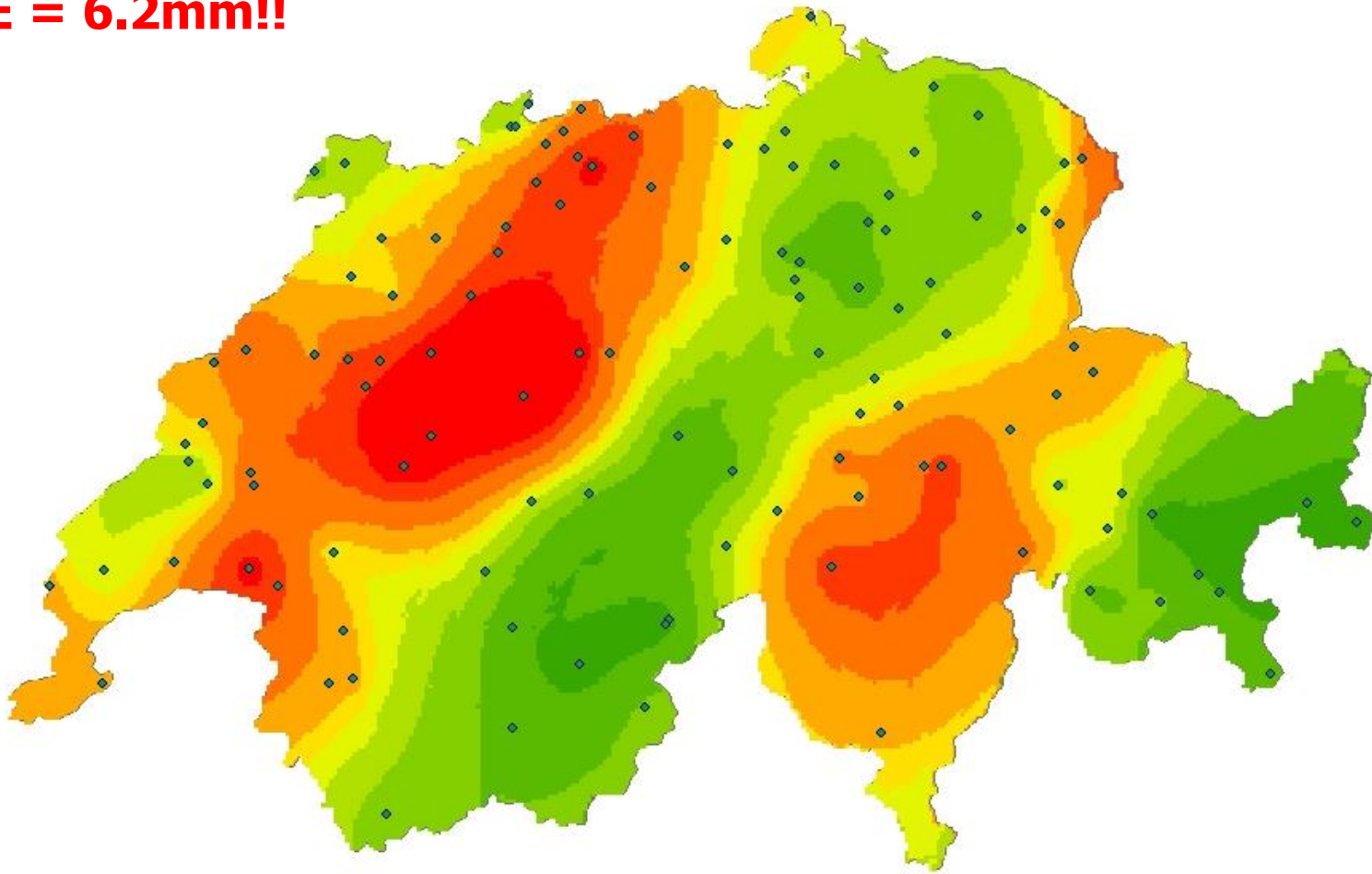


4. **Visualise** and **qualitatively assess** results (Don't forget this stage – **do the results make sense?**)



Kriging example

RMSE = 6.2mm!!



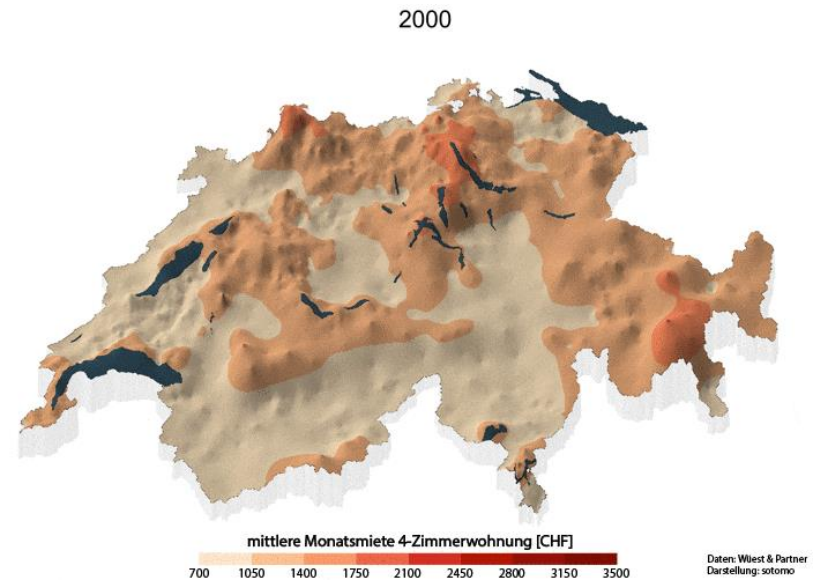
Ordinary kriging: Spherical model, lag = 30km; nugget~0 ; range = 78km; sill ~ 13

More on geostatistics

- The **semivariogram** can help us to **improve sampling strategies**
- These techniques also allow us to **estimate errors**
- In general, use of **geostatistics** is **non-trivial** and requires considerable further study
- Before applying such techniques you **must** understand the basic principles and underlying hypotheses described above...

Back to our rent prices

- Original data were values for **Swiss Gemeinde**
- These were assigned to **centroid** of each **Gemeinde**
- **Spatial interpolation** method was **ordinary kriging** with an **exponential model** - **exact, local** interpolation
- **Temporal interpolation** is linear
- Interpolation performed with [R](#)



Many thanks to Mario Nowak
for these details

Summary

- **Spatial interpolation** assumes that **spatial autocorrelation** is present
- **Wide range** of methods exist – you should understand the basic assumptions of the method you apply
- In our example, **best results** were for **ordinary kriging**, followed by **IDW** with “default” parameters (in terms of **RMSE**)
- However, **most techniques** gave similar results – other, **qualitative**, features of the **surfaces** should also be considered
- **Geostatistical techniques** use **theory** to help us identify appropriate parameters – but basic interpolation methods remain similar

Some of this material is hard – you must do some reading and thinking...

Next week

- We will look at how we can work with **terrain models**
- In particular we will look at calculating so-called **primary and secondary terrain indices** with physical meanings

References

- Key reading: Burrough et al. (Chapters 8 and 9)
- Also very useful: Geographic Information Analysis (O'Sullivan and Unwin, 2003) -> Chapters 8 and 9